In this solution, unit system $k_B = 1$ is used. One can restore it by dimensional analysis $\beta = \frac{1}{T}$,

7.1 3D→2D condensation of ideal gas

(a)
The particle density at condensed surface can be expressed as

$$n_{cond} = \frac{1}{(2\pi\hbar)^2}\int d^2p_{2D}e^{-\beta(\frac{p_{2D}^2}{2m}-\Delta-\mu_{2D})} = \frac{1}{(2\pi\hbar)^2}\frac{2m\pi}{\beta}e^{\beta(\Delta+\mu_{2D})} = \frac{1}{\lambda^2}e^{\beta(\mu_{2D}+\Delta)} \tag{1}$$

where $\lambda = \sqrt{\frac{2\pi\hbar^2\beta}{m}}$. Then we want to calculate the partition function for single particle of 3D part

$$\begin{aligned} z_{3D} &= \frac{1}{(2\pi\hbar)^3}\int d^3xd^3p_{3D}e^{-\beta(\frac{p_{3D}^2}{2m})} = \frac{V}{(2\pi\hbar)^3}\left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}} = \frac{V}{\lambda^3} \\ \mathcal{Z}_{3D} &= \sum_{N=0}^{\infty}\frac{1}{N!}z_{3D}^N e^{N\beta\mu_{3D}} = \exp\left[e^{\beta\mu_{3D}}z_{3D}\right] \\ \Omega_{3D} &= -\frac{1}{\beta}\log\mathcal{Z}_{3D} = -\frac{e^{\beta\mu_{3D}}z_{3D}}{\beta} \\ p &= -\left(\frac{\partial\Omega(T,V,\mu)}{\partial V}\right)_{T,\mu} = \frac{1}{\beta}\frac{1}{\lambda^3}e^{\beta\mu_{3D}} \end{aligned} \tag{2}$$

In co-existing phase $\mu_{2D} = \mu_{3D}$, and

$$\Rightarrow p = \frac{1}{\beta\lambda^3}n_{cond}\lambda^2e^{-\beta\Delta} = \frac{n_{cond}}{\beta\lambda}e^{-\beta\Delta} \tag{3}$$

(b)
Use Clausius-Clapeyron equation, and ideal gas law $pv_{gas} = T$

$$l = T(v_{gas} - v_{liquid})\frac{\partial p}{\partial T} = Tv_{gas}n_{cond}\left(\frac{m}{2\pi\hbar^2}\right)e^{-\beta\Delta}\left[\frac{3}{2}T^{\frac{1}{2}} + T^{\frac{3}{2}}\frac{\Delta}{T^2}\right] = v_{gas}p\left[\frac{3}{2}+\frac{\Delta}{T}\right] = \Delta + \frac{3}{2}T \tag{4}$$

7.2 Boiling temperature of water

Barometric formula for pressure is $p(h) = p(h=0)\exp(-mgh/T_{\rm air})$. By Clapeyron-Clausius equation at sea level

$$\begin{aligned} \frac{\partial p}{\partial T} &= \frac{L}{T(v_{\rm air}-v_{\rm liquid})} \simeq \frac{L}{Tv_{\rm air}} = \frac{Lp}{T^2} \\ \Rightarrow \frac{dp}{p} &= L\frac{dT}{T^2} \\ \Rightarrow p(T) &= p(T=T_{\rm boiling})e^{-\frac{L}{T}+\frac{L}{T_{\rm boiling}}} \end{aligned} \tag{5}$$

At sea level, we have $T = T_{\rm boiling}$, $h = 0$. $p(T=T_{\rm boiling}) = p(h=0) \equiv p_0$. We find

$$\begin{aligned} -\frac{L}{T(h)} + \frac{L}{T_{\rm boiling}} = -\frac{mgh}{T_{\rm air}} &\Rightarrow \frac{1}{T(h)} = \frac{mgh}{L}\frac{1}{T_{\rm air}} + \frac{1}{\text{boiling}} \\ \frac{1}{T(h=1\text{ mile})} = \frac{29\times9.8\times1.6}{4\times10^4\times290} + \frac{1}{373} &\Rightarrow T(h=1\text{ mile}) = 367.65K \end{aligned} \tag{6}$$

So we get $\Delta T = 373.15 - 367.65 = 5.5K = 5.5°C = 9.9°F$.

7.3 3-spin Ising model

states of the system:

$$\begin{aligned} &(+,+,+): 1 \text{ state; } E = -3J - 3H \\ &(+,+,-): 3 \text{ states; } E = J - H \\ &(+,-,-): 3 \text{ states; } E = J + H \\ &(-,-,-): 1 \text{ state; } E = -3J + 3H \end{aligned} \tag{7}$$

Partition function:

$$Z = \sum_{\text{states}} e^{-\beta E} = e^{3\beta(J+H)} + 3e^{-\beta(J-H)} + 3e^{-\beta(J+H)} + e^{3\beta(J-H)} \tag{8}$$

Magnetization

$$\begin{aligned} M &= \frac{1}{\beta}\frac{\partial}{\partial H}\log Z = \frac{3e^{3\beta(J+H)} + 3e^{-\beta(J-H)} - 3e^{-\beta(J+H)} - 3e^{3\beta(J-H)}}{Z} \\ M(H=0) &= \frac{3e^{3\beta J} + 3e^{-\beta J} - 3e^{-\beta J} - 3e^{3\beta J}}{e^{3\beta J} + 3e^{-\beta J} + 3e^{-\beta J} + e^{3\beta J}} = 0 \end{aligned} \tag{9}$$

In high temperature limit $\beta \to 0$

$$\begin{aligned} M &\sim 3\frac{(1+3\beta(J+H)) + (1-\beta(J-H)) - (1-\beta(J-H)) - (1+3\beta(J-H))}{(1+3\beta(J+H)) + 3(1-\beta(J-H)) + 3(1-\beta(J+H)) + (1+3\beta(J-H))} \\ &= 3\frac{8\beta H}{8} = 3\beta H \\ \chi &= \frac{\partial M}{\partial H} = 3\beta \end{aligned} \tag{10}$$

Energy at high T is

$$\begin{aligned} E &= -\frac{\partial}{\partial \beta}\log Z = -3\frac{(J+H)e^{3\beta(J+H)} + (H-J)e^{\beta(H-J)} - (J+H)e^{-\beta(J+H)} + (J-H)e^{3\beta(J-H)}}{Z} \\ E(\beta \to 0) &= -3\beta\frac{3(J+H)^2 + (H-J)^2 + (J+H)^2 + 3(J-H)^2}{8} = -3\beta(J^2+H^2) \\ C &= \frac{\partial E}{\partial T} = -\beta^2\frac{\partial E}{\partial \beta} = 3\beta^2(J^2+H^2) \end{aligned} \tag{11}$$