In this solution, unit system $k_B = 1$ is used. One can restore it by dimensional analysis $\beta = \frac{1}{T}$,

6.1 Bose condensation in harmonic trap

For harmonic potential, the energy is

$$\varepsilon = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2\vec{r}^2 \tag{1}$$

At critical temperature β_C, $\mu(\beta_C) = 0$.
If the system is classical:

$$N = \frac{1}{(2\pi\hbar)^3}\int d^3x d^3p \frac{g}{e^{-\beta_C\varepsilon} - 1} = \frac{(4\pi)^2}{(2\pi\hbar)^3}\int drdp \frac{gr^2p^2}{e^{\beta_C(\frac{p^2}{2m} + \frac{m\omega^2}{2}r^2)} - 1}\ ; \tag{2}$$

We change to another set of variables $u^2\cos^2\theta = \frac{p^2}{2m}$, $u^2\sin^2\theta = \frac{m\omega^2}{2}r^2$

$$\begin{aligned} N &= g\frac{(4\pi)^2}{(2\pi\hbar)^3}(2m)^{\frac{3}{2}}\left(\frac{2}{m\omega^2}\right)^{\frac{3}{2}}\int \frac{u^5\cos^2\theta\sin^2\theta}{e^{\beta_C u^2} - 1}dud\theta\ ;\ x = \beta_C u^2 \\ &= g\frac{16}{\pi(\hbar\omega)^3}\frac{1}{2\beta_C^3}\int\frac{x^2dx}{e^x - 1}\frac{1}{4}\int_0^{\frac{\pi}{2}} d\theta\sin^2 2\theta \\ &= g\frac{8}{\pi(\beta_C\hbar\omega)^3}I_3\frac{\pi}{16} \\ &= \frac{gI_3}{2}\frac{1}{(\beta_C\hbar\omega)^3} \\ \Rightarrow T_C &= \left(\frac{2N}{gI_3}\right)^{\frac{1}{3}}\hbar\omega \end{aligned} \tag{3}$$

If the system is quantum, $E_{\vec{n}} = \hbar\omega(n_x + n_y + n_z + \frac{3}{2})$, We have to be careful ground state energy $E_g = E_{\vec{0}} = \frac{3}{2}\hbar\omega$. In usual Bose-Einstein condensation calculation, we assume $\mu(T_C) = E_g$. However by doing so, the summation of

$$N = \sum_{\text{states}}\frac{g}{e^{\beta(E-\mu)} - 1} = \sum_{\vec{n}}\frac{g}{e^{b(n_x+n_y+n_z)} - 1} = \infty + \sum_{\vec{n}\neq 0}\frac{g}{e^{b(n_x+n_y+n_z)} - 1} = \infty \tag{4}$$

The infinity comes from the ground state $\frac{1}{e^0 - 1} = \frac{1}{1-1} = \infty$. Equation $N = \infty$ gives us no information. One needs to regulate and renormalize this formula. A way to do so is to put back ground state energy, i.e. set $\mu = 0$.

$$N = \sum_{\vec{n}}\frac{g}{e^{b(n_x+n_y+n_z+\frac{3}{2})} - 1} = \sum_{k=1}\sum_{\vec{n}} e^{-kb(n_x+n_y+n_z+\frac{3}{2})} = g\sum_{k=1}\left[\frac{e^{\frac{-bk}{2}}}{1 - e^{-bk}}\right]^3 = g\sum_{k=1}\left[\frac{1}{2\sinh\frac{bk}{2}}\right]^2 \tag{5}$$

We can evaluate its upper bound by the fact $2\sinh\frac{bk}{2} \geq bk\ \forall k \geq 0$.

$$N \leq g\sum_{k=1}\frac{1}{(bk)^3} = \frac{g}{b^3}\sum_{k=1}\frac{1}{k^3} = \frac{g\zeta(3)}{b^3} \tag{6}$$

In thermodynamics/semi-classical limit $b \to 0$, $N \to \frac{g\zeta(3)}{b^3}$, which matches our classical calculation. ($I_3 = \Gamma(3)\zeta(3) = 2\zeta(3)$). One can try to do full quantum calculation by evaluating the summation in eq.(5)

A common mistake here is defining $n = n_x + n_y + n_z$ and then replace the summation in eq.(4) with $\int d^3n = \int 4\pi n^2 dn$. However neither n^2 nor $\frac{n^2}{6}$ (some people write this factor, I don't know why.) are the correct degenercy factor, e.g. $n = 1$ has degenercy 3, $(n_x, n_y, n_z) = (1,0,0)$, $(0,1,0)$, $(0,0,1)$, $1^2 = 1 \neq 3$. The right way is replacing $\sum$ with $\int g(n)dn$, $g(n) = \frac{(n+1)(n+2)}{2}$ (one may check this is the correct degenercy factor). One may immediately found the integration becomes

$$\frac{g}{2}\int_0^\infty \frac{n^2+3n+2}{e^{bn}-1}dn = \frac{I_3}{2b^3} + \frac{3I_2}{b^2} + \frac{I_1}{b} \; ; \; I_1 = \Gamma(1)\zeta(1) = \infty \tag{7}$$

Eventually one still got $N = \infty + \ldots$ if all calculations are done correctly, which tells you nothing.

6.2 Temperature of Earth

(a)
First we calculate the energy absorbed by Earth

$$\begin{aligned} \text{Energy} &= \text{ energy flux} \times \text{ cross section} \\ &= \sigma T_S^4 \times \frac{4\pi R_S^2}{4\pi d^2} \times \pi R_E^2 \\ &= \sigma T_E^4 \times 4\pi R_E^2 \\ \Rightarrow T_E &= \sqrt{\frac{R_S}{2d}} T_S = 289K \end{aligned} \tag{8}$$

(b)
Assume moon has no surface heat transfer, a patch with area A at angle $0 < \theta < \frac{\pi}{2}$. (z-direction poining toward sun)

$$\begin{aligned} \sigma T_m(\theta)^4 \times A &= \sigma T_S^4 \times \frac{4\pi R_S^2}{4\pi d^2} \times \cos\theta \times A \\ \Rightarrow T_m(\theta) &= (\cos\theta)^{\frac{1}{4}} \sqrt{\frac{R_S}{d}} T_E = (4\cos\theta)^{\frac{1}{4}} T_E \end{aligned} \tag{9}$$

Maximum $T_m(0) = 409K$, minimum $T_m(\frac{\pi}{2}) = 0$.

6.3 Voltage fluctuation in an LC circuit

We can write down Hamiltonian

$$H = \frac{1}{2}LI^2 + \frac{Q^2}{2C} = \frac{\Pi^2}{2L} + \frac{1}{2}L\omega^2 Q^2 \; ; \; \Pi \equiv LI = L\dot{Q} \tag{10}$$

We can view Π as "conjugate momentum" of "generalized coordinate" Q. $Q\Pi$ has dimension as angular momentum.
If the system is classical, the partition function can be found by

$$\begin{aligned} Z &= \sum_{\text{state}} e^{-\beta\varepsilon} = \int \frac{d\Pi dQ}{2\pi\hbar} e^{-\frac{\beta}{2}(\frac{\Pi^2}{L} + L\omega^2 Q^2)} = \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi L}{\beta}}\sqrt{\frac{2\pi}{\beta L\omega}} = \frac{1}{\beta\hbar\omega} \\ \overline{V^2} &= \frac{\sum V^2 e^{-\beta\varepsilon}}{Z} = \frac{1}{C^2}\frac{\sum Q^2 e^{-\beta\varepsilon}}{Z} = \frac{1}{C^2}\frac{-2}{\beta L}\frac{\partial}{\partial\omega^2}\log Z = \frac{1}{C^2\beta L\omega^2} = \frac{1}{C\beta} \\ \delta V &= \sqrt{\overline{V^2}} = \sqrt{\frac{T}{C}} \end{aligned} \tag{11}$$

An easy way to check this result is by considering the energy held by capacitor $E_C = \frac{1}{2}CV^2 = E_T = \frac{1}{2}T \Rightarrow V^2 = \frac{T}{C}$
If the system is quantum, set quantization condition as $[Q, \Pi] = i\hbar$. Defining $a = \sqrt{\frac{L\omega}{2\hbar}}\left(Q + i\frac{\Pi}{L\omega}\right)$, $a^\dagger = \sqrt{\frac{L\omega}{2\hbar}}\left(Q - i\frac{\Pi}{L\omega}\right)$, we obtain

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) \Rightarrow E_n = \hbar\omega(n + \frac{1}{2}) \tag{12}$$

Calculate partition function

$$
\begin{aligned}
Z &= \sum_{\text{state}} e^{-\beta\varepsilon} = \sum_{n=0} e^{-\beta\hbar\omega(n+\frac{1}{2})} = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{2\sinh\frac{1}{2}\beta\hbar\omega} \\
\overline{V^2} &= \frac{-1}{C^2\beta L\omega}\frac{\partial}{\partial\omega}\log Z = \frac{\omega}{C\beta}\frac{1}{2}\beta\hbar\coth\frac{1}{2}\beta\hbar\omega = \frac{\hbar\omega}{2C}\coth\frac{1}{2}\beta\hbar\omega \\
\delta V &= \sqrt{\overline{V^2}} = \sqrt{\frac{\hbar\omega}{2C}\coth\frac{1}{2}\beta\hbar\omega}
\end{aligned}
\tag{13}
$$

As $\hbar \to 0$, classical solution is obtained.