

In this solution, we use $\beta = \frac{1}{T}$ for the inverse temperature and $g_e = 2$ for the electron spin degeneracy which we will often keep as g_e to keep track of it.

5.1 Electron bands in a semiconductor

Non-degenerate ($g_e = 2$) semiconductor at room temperature $T \ll \Delta$, $\mu \gg T$, $(\Delta - \mu) \gg T$. (a)
The electron gas is assumed to be ideal. Since $T \ll \mu$, $|\Delta - \mu|$, the gas in permitted energy bands is approximately Boltzmann-like, i.e. in the conduction band $\varepsilon_C = \frac{p^2}{2m_C} + \Delta$

$$\begin{aligned} n_e &= \frac{N_e}{V} = g_e \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{(\varepsilon_C - \mu)/T} + 1} = \frac{g_e}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{e^{p^2/(2m_C T) + (\Delta - \mu)/T} + 1} \\ &\simeq \frac{g_e}{(2\pi\hbar)^3} e^{-(\Delta - \mu)/T} \int_0^\infty dp 4\pi p^2 e^{-p^2/(2m_C T)} \\ &= \frac{g_e}{(2\pi\hbar)^3} e^{-(\Delta - \mu)/T} (2\pi m_C T)^{3/2} = g_e \left(\frac{m_C T}{2\pi\hbar^2} \right)^{3/2} e^{-(\Delta - \mu)/T} \end{aligned} \quad (1)$$

(the integral in the second line is the usual Gaussian integral in 3d similar to the Boltzmann ideal gas). For holes $\varepsilon_V = -p^2/(2m_V)$, and we use the complementary distribution $n_h = 1 - n_e$,

$$\begin{aligned} n_h &= N_h/V = g_e \int \frac{d^3p}{(2\pi\hbar)^3} \left[1 - \frac{1}{e^{(\varepsilon_V - \mu)/T} + 1} \right] \\ &= g_e \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{1 + e^{(\mu - \varepsilon_V)/T}} = \frac{g_e}{(2\pi\hbar)^3} \int_0^\infty \frac{4\pi p^2 dp}{1 + e^{p^2/(2m_V T) + \mu/T}} \\ &\simeq \frac{g_e}{(2\pi\hbar)^3} e^{-\mu/T} \int_0^\infty dp 4\pi p^2 e^{-p^2/(2m_V T)} \\ &= \frac{g_e}{(2\pi\hbar)^3} e^{-\mu/T} (2\pi m_V T)^{3/2} = g_e \left(\frac{m_V T}{2\pi\hbar^2} \right)^{3/2} e^{-\mu/T} \end{aligned} \quad (2)$$

(a) To calculate the equilibrium densities of electrons and holes $n_e = n_h = n_i$, we can use the following trick:

$$n_i = \sqrt{n_e n_h} = \sqrt{g_e^2 \left(\frac{m_C T}{2\pi\hbar^2} \right)^{3/2} \left(\frac{m_V T}{2\pi\hbar^2} \right)^{3/2} e^{-\Delta/T}} = g_e \left(\frac{m^* T}{2\pi\hbar^2} \right)^{3/2} e^{-\Delta/(2T)} = n_e = n_h \quad (3)$$

where $m^* = \sqrt{m_C m_V}$.

(b) The chemical potential then is easy to find, for example, from the density of the holes:

$$e^{-\mu/T} = \frac{n_h}{g_e} \left(\frac{2\pi\hbar^2}{m_V T} \right)^{3/2} = \left(\frac{m^*}{m_V} \right)^{3/2} e^{-\Delta/(2T)} \Rightarrow \mu = \frac{\Delta}{2} - \frac{3T}{2} \log \frac{m^*}{m_V} = \frac{\Delta}{2} + \frac{3T}{4} \log \frac{m_V}{m_C} \quad (4)$$

5.2 Thermocouple

Using the Sommerfeld trick, in the lecture we have derived that the chemical potential for a degenerate Fermi gas is shifted from the Fermi energy $\varepsilon_F = \mu(T = 0)$ by $O(T^2)$:

$$\mu \approx \varepsilon_F - \frac{\pi^2 T^2}{6} \left(\frac{g'}{g} \right)_{\mu \approx \varepsilon_F} \quad (5)$$

In computing the last ratio, we can use the fact that $\mu \approx \varepsilon_F$ (the difference would lead to a higher-order correction in (T/ε_F)). Recalling that in 3 dimensions the density of states is $g(\varepsilon) \propto \varepsilon^{1/2}$ (non-relativistic

ideal gas), hence $g'(\varepsilon) = g(\varepsilon)/(2\varepsilon)$, and the chemical potential depends on the temperature as

$$\mu(T) = \varepsilon_F - \frac{\pi^2 T^2}{12\varepsilon_F} \quad (6)$$

Due to temperature gradient, there may be imbalance of chemical potential, and the electrons flow to the region with lower μ . The resulting charge density leads to non-uniform electrostatic potential $\varphi(x)$, and the electric field $\vec{E} = -\nabla\varphi$ compensates the imbalance of μ . Once the transient processes stop, and assuming that current drawn by the measurement circuit is negligible, the full chemical potential including the electric field is

$$\mu'(x) = \mu(T(x)) - e\varphi(x) = \text{const}, \quad (7)$$

(the electron charge is $(-e) < 0$), thus the electrostatic potential is

$$\varphi(x) = \varphi_0 + \frac{\pi^2 T^2}{12e} \quad (8)$$

where φ_0 is an irrelevant constant. For a two-segment circuit made of two different metals with the same temperature differential $T_{meas} - T_{ref}$ on each, the resulting *emf* is

$$\mathcal{E} = \Delta\varphi^{\text{Cu}} - \Delta\varphi^{\text{Al}} = \frac{\pi^2}{12e} \left(\frac{1}{\varepsilon_F^{\text{Cu}}} - \frac{1}{\varepsilon_F^{\text{Al}}} \right) (T_{meas}^2 - T_{ref}^2) \approx \frac{\pi^2 T}{6e} \left(\frac{1}{\varepsilon_F^{\text{Cu}}} - \frac{1}{\varepsilon_F^{\text{Al}}} \right) \Delta T \quad (9)$$

Thus the temperature dependent part of EMF in copper aluminum junction is

$$\left(\frac{\partial \mathcal{E}}{\partial T} \right)_{\text{Cu-Al}} = \frac{\mathcal{E}}{T_{meas} - T_{ref}} = \frac{\pi^2 T}{6e} \left(\frac{1}{\varepsilon_F^{\text{Cu}}} - \frac{1}{\varepsilon_F^{\text{Al}}} \right) \quad (10)$$

At room temperature $T \approx 300 \text{ K} = 2.59 \times 10^{-2} \text{ eV}$ (energy $1 \text{ eV} = 1.16 \times 10^4 \text{ K}$), we get

$$\left(\frac{\partial \mathcal{E}}{\partial T} \right)_{\text{Cu-Al}} \approx 2.1 \times 10^{-7} \text{ V/K}. \quad (11)$$

Note that this quantity is dimensionless in unified energy units with $k_B = 1$, $e = 1$, but conversion to physical units requires an extra factor $1\text{K}/1\text{eV} = 1/(1.16 \times 10^4)$.

5.3 Degenerate relativistic Fermi gas

Start with number density at $T = 0$

$$\begin{aligned} n &= g_e \int_{|p| \leq p_F} \frac{d^3 p}{(2\pi\hbar)^3} = \frac{g_e}{(2\pi\hbar c)^3} \int_0^{\varepsilon_F} d\varepsilon 4\pi\varepsilon^2 = g_e \frac{\frac{4\pi}{3}\varepsilon_F^3}{(2\pi\hbar)^3} \\ \Rightarrow g(\varepsilon) &= \frac{dn(\varepsilon_F)}{d\varepsilon_F} = g_e \frac{4\pi\varepsilon_F^2}{(2\pi\hbar)^3} = \frac{3n}{\varepsilon_F} \end{aligned} \quad (12)$$

The energy is obtained by integrating ε :

$$E = V g_e \int_{|p| \leq p_F} \frac{d^3 p}{(2\pi\hbar)^3} \varepsilon = V \frac{g_e}{(2\pi\hbar c)^3} \int_0^{\varepsilon_F} d\varepsilon 4\pi\varepsilon^3 = V g_e \frac{\frac{\pi}{2}\varepsilon_F^4}{(2\pi\hbar c)^3} = V \frac{3n}{4} \varepsilon_F = \frac{3N}{4} \varepsilon_F \quad (13)$$

Note that the Fermi energy ε_F scales with the density $n = N/V$ as $\varepsilon_F \propto n^{1/3} \propto V^{-1/3}$, hence its derivative $d\varepsilon_F/dV = -\varepsilon_F/(3V)$. To calculate the pressure at $T = 0$, we take the derivative

$$\begin{aligned} P &= \lim_{T \rightarrow 0} T \frac{\partial}{\partial V} \log Z = - \lim_{T \rightarrow 0} \frac{\partial}{\partial V} F = - \lim_{T \rightarrow 0} \frac{\partial}{\partial V} (E - TS) = - \left(\frac{\partial E}{\partial V} \right)_{T=0} \\ &= - \frac{3N}{4} \frac{\partial \varepsilon_F}{\partial V} = - \frac{E}{\varepsilon_F} \left(\frac{-\varepsilon_F}{3V} \right) = \frac{E}{3V} \end{aligned} \quad (14)$$

Note that at zero temperature the entropy of the degenerate Fermi gas is zero, and one can calculate the pressure as the derivative at either constant T or S : $P = - \left(\frac{\partial E}{\partial V} \right)_S = - \left(\frac{\partial E}{\partial V} \right)_T$.

5.4 Paramagnetism of a degenerate Fermi gas

One can consider electrons with spin \pm as separate gases with densities n_{\pm} that are at equilibrium and thus have the same chemical potential μ_0 ,

$$\mu_0 = \varepsilon_F^+ - \mu H = \varepsilon_F^- + \mu H \quad (15)$$

Their densities n_{\pm} can be computed from their Fermi energies ε_F^{\pm} ,

$$n_{\pm} = \frac{1}{2}n(\varepsilon^{\pm}) = \frac{1}{2}n(\mu_0 \pm \mu H) \approx \frac{1}{2}n(\mu_0) \pm \frac{1}{2}g(\mu_0)(\mu H) \quad (16)$$

where $n(\varepsilon)$ and $g(\varepsilon) = dn/d\varepsilon$ are the spatial density and the density of states of free electrons (including the spin degeneracy). Therefore, the polarization of the Fermi sea is $\Delta n = n_+ - n_-$ and the magnetization is $M = \mu\Delta n = \mu^2 H g(\varepsilon_F)$. Taking $\mu_0 \approx \varepsilon_F$ is OK for degenerate Fermi gases in weak fields.

It is convenient to express results as functions of both n and ε_F to avoid nasty numerical factors:

(a) for a non-relativistic Fermi gas, $\varepsilon_F = \frac{p_F^2}{2m}$ and $n(\varepsilon_F) \propto \varepsilon_F^{3/2}$, hence $g(\varepsilon_F) = dn(\varepsilon_F)/d\varepsilon = 3n/(2\varepsilon_F)$. The magnetization is

$$M = \mu^2 H g(\varepsilon_F) = \frac{3\mu^2 H n}{2\varepsilon_F}; \quad (17)$$

(b) for an ultrarelativistic gas, $\varepsilon_F = cp_F$ and $n(\varepsilon_F) \propto \varepsilon_F^3$, hence $g(\varepsilon_F) = dn(\varepsilon_F)/d\varepsilon_F = 3n/\varepsilon_F$

$$M = \mu^2 H g(\varepsilon_F) = \frac{3\mu^2 H n}{\varepsilon_F} \quad (18)$$

One can express either n or ε_F using the other variable to simplify the results, i.e. $g(\varepsilon_F) = g_e \frac{m p_F}{2\pi^2 \hbar^3} = g_e \frac{m^{3/2} \sqrt{\varepsilon_F}}{\sqrt{2}\pi^2 \hbar^3}$ for a non-relativistic gas. However, this is hardly any more illuminating, and is not required.