

In this solution, unit system k_B is used. One can restore it by dimensional analysis $\beta = \frac{1}{T}$,

4.1 Gas leak hole

(a)

Since $A \ll \lambda^2$, we may assume the gas near the hole is in thermal equilibrium. By Maxwell distribution $f(v)$ and flux for particular speed passing through the hole Φ , (denote $b = \frac{\beta m}{2}$):

$$f(v) = \left[\sqrt{\frac{b}{\pi}} \right]^3 4\pi v^2 e^{-bv^2} \quad (1)$$

To find the flux of particle moving in volicity v passing through the hole, we may consider in solid angle $\frac{d\Omega}{4\pi}$, there are $\frac{d\Omega}{4\pi} \rho v \cos \theta$ particles hitting in small area A on the surface of box.

$$\Phi = \int d\Phi = \int \rho v \cos \theta \frac{d\Omega}{4\pi} = \frac{1}{4} \rho v \quad (2)$$

Thus the energy inside the jet is

$$\langle E \rangle_{\text{jet}} = \frac{\int E \Phi f(v) d^3v}{\int \Phi f(v) d^3v} = \frac{m \int v^5 e^{-bv^2} dv}{2 \int v^3 e^{-bv^2} dv} = \frac{m \int x^2 e^{-bx} dx}{2 \int x e^{-bx} dx} = \frac{m}{2} \frac{\frac{\partial^2}{\partial b^2} \frac{1}{b}}{-\frac{\partial}{\partial b} \frac{1}{b}} = \frac{m}{2} \frac{2b^2}{b^3} = 2T \quad (3)$$

(b)

We may assume the hole is at top of container. Net thrust is on the z-direction. The force contributed by a bounch of particle moving in velocity v from solid angle $d\Omega$ is

$$\begin{aligned} F_z &= \frac{dp_z}{dt} = m \frac{dN_{\text{jet}}}{dt} v_z = m \rho \cos \theta A v \times v \cos \theta = m \rho A v^2 \cos^2 \theta \\ \langle F_z \rangle_{\text{tank}} &= \frac{\int F_z \frac{d\Omega}{4\pi} f(v) dv}{\int f(v) d^3v} = m \rho A \frac{2\pi \int \cos^2 \theta d \cos \theta \int v^4 e^{-bv^2} dv}{4\pi \int v^2 e^{-bv^2} dv} = m \rho A \frac{1}{6} \frac{\frac{\partial^2}{\partial b^2} \sqrt{\frac{\pi}{b}}}{-\frac{\partial}{\partial b} \sqrt{\frac{\pi}{b}}} \\ &= m \rho A \frac{1}{6} \frac{\frac{3}{2} \frac{1}{2} \sqrt{\frac{1}{b^5}}}{\frac{1}{2} \sqrt{\frac{1}{b^3}}} = \frac{m \rho A}{4b} = \frac{\rho A T}{2} \end{aligned} \quad (4)$$

Alternatively, we may also focus on the jet itself. We put a hemisphere dome on top of the hole, then the force acting on the dome is on radical direction F_r . and we know $F_z = \int F_r \cos \theta d \cos \theta = \frac{F_r}{2}$

$$\langle F_z \rangle_{\text{jet}} = \frac{1}{2} \langle F_r \rangle_{\text{jet}} = \frac{1}{2} \left\langle m \frac{dN_{\text{jet}}}{dt} v \right\rangle_{\text{jet}} = \frac{m \rho A}{8} \langle v^2 \rangle_{\text{jet}} = \frac{m \rho A}{8} \frac{4T}{m} = \frac{\rho A T}{2} \quad (5)$$

There is an easy way to find this answer: microscopically, pressure is generated by particle hitting on container then bouncing back. The momentum change drops to a half when the particle travels into the hole. Leading pressure drops also by a factor of half $P' = \frac{1}{2} P$. Thus $F = P' A = \frac{1}{2} P A = \frac{1}{2} \frac{N T A}{V}$. There are two common mistake one may take when doing this question. First, the average velocity square does not equal to the average of squared velocity, i.e. $\langle v \rangle^2 \neq \langle v^2 \rangle$. This is because of the integration we took along with distribution function. That means, when one calculates $\langle \frac{dN}{dt} \rangle = \frac{1}{4} \rho A \langle v_z \rangle$, $\langle v_z \rangle \neq \sqrt{\frac{2E}{3m}}$.

The second mistake immediately follows when assuming $v_x^2 = v_y^2 = v_z^2 = \frac{v^2}{3}$. Gas inside the tank obeys relation because of the spacial rotation symmery in \mathbb{R}^3 . However, jet picks specific direction, leading the break down of this spacial rotation symmetry. We can calculate the average squared velocities in each direction (assume the hole is on top of the tank, jet is in z-direction.)

$$\begin{aligned} \langle v_z^2 \rangle &= \int v_z^2 d\Phi f(v) dv \propto 2\pi \int \cos^3 \theta d \cos \theta = \frac{\pi}{2} \\ \langle v_x^2 \rangle &= \int v_x^2 d\Phi f(v) dv \propto \pi \int \sin^3 \theta \cos \theta d\theta = \frac{\pi}{4} = \langle v_y^2 \rangle \end{aligned} \quad (6)$$

As we can see, the mean squared velocity in z-direction is 2 times larger than x and y directions. This also agrees with part (a) that we have average energy in jet as $2T$ instead of $\frac{3}{2}T$.

(c)

Assuming the gas is monotonic and the leak is slow. Assume initially the gas inside box has N particles and temperature T . There are dN particles leaking out, carrying energy $dE = 2TdN$, and temperature drops by dT . By energy conservation

$$\frac{3}{2}NT = 2TdN + \frac{3}{2}(N - dN)(T - dT) \Rightarrow \frac{1}{2}TdN - \frac{3}{2}NdT = 0 \quad (7)$$

$$\begin{aligned} \frac{dT}{dt} &= \left\langle \frac{T}{3N} \frac{dN}{dt} \right\rangle_{\text{tank}} = \left\langle \frac{T}{3N} A\Phi \right\rangle_{\text{tank}} = \frac{AT}{3N} \langle \Phi \rangle_{\text{tank}} \\ &= -\frac{AT\rho}{12N} \frac{\int v f(v) dv}{\int f(v) dv} = -\frac{AT}{6V} \frac{1}{\sqrt{\pi b}} \equiv \alpha T^{\frac{3}{2}}; \quad \alpha = \frac{A}{6V} \sqrt{\frac{2}{m\pi}} \end{aligned} \quad (8)$$

Alternatively, one can use time derivative of energy. $E = Fx \Rightarrow \frac{dE}{dt} = \frac{dF}{dt}x + Fv \simeq Fv$, by assuming the leaking is slow, we may neglect time derivative of force.

$$\begin{aligned} \frac{3}{2} \left[N_{\text{tank}} \frac{dT}{dt} + T \frac{dN_{\text{tank}}}{dt} \right] &= \frac{dE}{dt} = -Fv = -\frac{\rho ATv}{2}; \quad \frac{dN_{\text{tank}}}{dt} = -\frac{dN_{\text{jet}}}{dt} = -\frac{1}{4}\rho Av \\ \frac{dT}{dt} &= \frac{2}{3N} \left[-\frac{\rho ATv}{2} + \frac{3}{2} \frac{\rho ATv}{4} \right] = \frac{ATv}{12V} = -\frac{AT}{6V} \frac{1}{\sqrt{\pi b}} = \alpha T^{\frac{3}{2}} \end{aligned} \quad (9)$$

Solve this ODE by integration with initial condition $T(t=0) = T_0$.

$$\alpha t + \text{const} = 2 \frac{1}{\sqrt{T}} \Rightarrow T(t) = \frac{4}{\left(\alpha t + \frac{2}{\sqrt{T_0}} \right)^2} \quad (10)$$

Since the question did not specify the equation of change of the gas temperature should depend on, or its form. You will get full score once you wrote down expressions of either $\frac{dT}{dN}$, or $\frac{dT}{dt}$.

4.2 Chemical equilibrium

For chemical reaction equilibrium $\delta G_{\text{tot}} = 0$. $1 + 2 \rightarrow 3$. Gases are ideal so we can assume μ_i is independent of other species. $\Rightarrow \mu_j = g_j$, with $\frac{\hbar^2}{2I} \ll T \ll \hbar\omega_3$. Vibration mode is 'frozen' and will not be excited by thermal energy. Thus we don't need to include vibration contribution.

$$G_{\text{tot}} = \sum_j n_j \mu_j = \sum_j n_j g_j \Rightarrow \sum_j \delta n_j g_j = 0 \Rightarrow \mu_1 + \mu_2 = \mu_3 \quad (11)$$

$$\begin{aligned} \mu_1 &= T \left[\log n_1 - \frac{3}{2} \log T - \frac{3}{2} \log \frac{m_1}{2\pi\hbar^2} \right] \\ \mu_2 &= T \left[\log n_2 - \frac{3}{2} \log T - \frac{3}{2} \log \frac{m_2}{2\pi\hbar^2} \right] \\ \mu_3 &= T \left[\log n_3 - \frac{5}{2} \log T - \frac{3}{2} \log \frac{m_3}{2\pi\hbar^2} - \log \frac{I}{\hbar^2} \right] \\ \Rightarrow \log n_3 &= \log n_1 + \log n_2 - \frac{1}{2} \log T - \frac{3}{2} \log \frac{m_1 m_2}{2\pi\hbar^2 m_3} + \log \frac{I}{\hbar^2} \\ \Rightarrow \frac{n_3}{n_1 n_2} &= K(T); \quad K(T) = \frac{1}{\sqrt{T}} \left[\frac{2\pi\hbar^2 m_3}{m_1 m_2} \right]^{\frac{3}{2}} \frac{I}{\hbar^2}; \quad (\text{Le Chatelier's principle}) \end{aligned} \quad (12)$$

Now suppose initially we have n_1 and n_2 number density of gas 1 and 2. When reaching equilibrium after chemical reaction, we have

$$(n_1 - n_3)(n_2 - n_3)K(T) = n_3 \Rightarrow n_3^2 - (n_1 + n_2 - \frac{1}{K(T)})n_3 + n_1n_2 = 0$$

$$n_3 = \frac{n_1 + n_2 - \frac{1}{K(T)} \pm \sqrt{(n_1 + n_2 - \frac{1}{K(T)})^2 - 4n_1n_2}}{2}$$
(13)

4.3 Debye model of crystal vibration

the definition of Debye temperature $T_D = \hbar\omega_{\max}$, where ω_{\max} is the maximal angular vibration frequency of crystal (phonon). To find ω_{\max} , we have number of phonon $N_{\text{phonon}} = 3N = 6 \simeq 3N$. By QM particle in the box, we have $p = \hbar k = \hbar \frac{\omega}{u}$

$$3N = \int g(n)d^3n = \int g(n) \frac{d^3x d^3p}{(2\pi\hbar)^3} = \frac{V}{(2\pi\hbar)^3} \int g(\omega)(\hbar k)^3 d^3k = \frac{V}{(2\pi)^2} \int_0^{\omega_{\max}} g(\omega) \left[\frac{2}{u_T^3} + \frac{1}{u_L^3} \right] 4\pi\omega^2 d\omega$$
(14)

$g(\omega)$ is degeneracy. $g(\omega) \simeq 3$ for 1 longitudinal and 2 transverse waves. So we get

$$3N = \frac{V}{2\pi^2} \left[\frac{2}{u_T^3} + \frac{1}{u_L^3} \right] \frac{\omega_{\max}^3}{3} \Rightarrow \omega_{\max} = \left[\frac{18\pi^2 N}{V \left(\frac{2}{u_T^3} + \frac{1}{u_L^3} \right)} \right]^{\frac{1}{3}} = \left[\frac{72\pi^2}{a^3 \left(\frac{2}{u_T^3} + \frac{1}{u_L^3} \right)} \right]^{\frac{1}{3}}$$
(15)

So one get

$$T_D = \frac{\hbar}{k_B} \left[\frac{72\pi^2}{a^3 \left(\frac{2}{u_T^3} + \frac{1}{u_L^3} \right)} \right]^{\frac{1}{3}}$$
(16)

We can summarize our result in table

Material	Debye Temperature	Debye Temperature(ref)
Aluminium	400 K	390 K
Copper	340 K	310 K
Lead	76 K	87 K

Now we can find heat capacity, one has

$$E = g \int_0^{\omega_{\max}} \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} d^3n$$

$$= \frac{4\pi V}{(2\pi)^3} \left[\frac{2}{u_T^3} + \frac{1}{u_L^3} \right] \int_0^{\omega_{\max}} \frac{\hbar\omega^3}{e^{\beta\hbar\omega} - 1} d\omega$$

$$= \frac{9N}{\beta} \frac{1}{(\beta\hbar\omega_{\max})^3} \int_0^{\beta\hbar\omega_{\max}} \frac{\xi^3 d\xi}{e^\xi - 1}$$
(17)

$$\frac{c_V}{\mu} = \frac{1}{N\mu} \frac{\partial E}{\partial T} = \frac{-\beta^2}{N\mu} \frac{\partial E}{\partial \beta}$$

$$= \frac{1}{\mu} \left[\frac{36}{(\beta T_D)^3} \int_0^{\beta T_D} \frac{\xi^3 d\xi}{e^\xi - 1} - \frac{9(\beta T_D)}{e^{\beta T_D} - 1} \right]$$

By using mathematica, we find

Material	lattice vibration contribution	full(ref)
Aluminium	0.844 J/(g*K)	0.904 J/(g*K)
Copper	0.367 J/(g*K)	0.384 J/(g*K)
Lead	0.120 J/(g*K)	0.127 J/(g*K)

4.4 Radiation heat transfer

(a)

Under thermal equilibrium, the absorption of a material must equal to its emission, i.e. the absorption rate must equal to emissivity. Thus we have reflection rate = 1-(absorption rate) = $1 - \epsilon$

(b)

On unit area the radiation acts on each radiative heat is

$$\begin{aligned}
 \mathcal{P}_1 &= \epsilon_1 \sigma T_1^4 + (1 - \epsilon_1) \mathcal{P}_2 \Rightarrow \Delta \mathcal{P} = \epsilon_1 \sigma T_1^4 - \epsilon_1 \mathcal{P}_2 \\
 \mathcal{P}_2 &= \epsilon_2 \sigma T_2^4 + (1 - \epsilon_2) \mathcal{P}_1 \Rightarrow -\Delta \mathcal{P} = \epsilon_2 \sigma T_2^4 - \epsilon_2 \mathcal{P}_1 \\
 &\Rightarrow (\epsilon_1 + \epsilon_2) \Delta \mathcal{P} = \epsilon_1 \epsilon_2 \sigma (T_1^4 - T_2^4) - \epsilon_1 \epsilon_2 \Delta \mathcal{P} \\
 &\Rightarrow \Delta \mathcal{P} = \frac{\epsilon_1 \epsilon_2 \sigma}{\epsilon_1 + \epsilon_2 - \epsilon_1 \epsilon_2} (T_1^4 - T_2^4)
 \end{aligned} \tag{18}$$

(c)

Since the system is in thermal equilibrium, the net radiation acting on the heat shield equals zero. That is $\Delta \mathcal{P}_L = \Delta \mathcal{P}_R$

$$\begin{aligned}
 \Delta \mathcal{P}_L &= \epsilon_1 \sigma (T_1^4 - T_{sh}^4) \\
 \Delta \mathcal{P}_R &= \epsilon_2 \sigma (T_{sh}^4 - T_2^4) \\
 &\Rightarrow [\epsilon_1 + \epsilon_2] T_{sh}^4 = \epsilon_1 T_1^4 + \epsilon_2 T_2^4
 \end{aligned} \tag{19}$$

$$T_{sh}^4 = \frac{\epsilon_1 T_1^4 + \epsilon_2 T_2^4}{\epsilon_1 + \epsilon_2} \Rightarrow T_{sh} = \left[\frac{\epsilon_1 T_1^4 + \epsilon_2 T_2^4}{\epsilon_1 + \epsilon_2} \right]^{\frac{1}{4}} \tag{20}$$

$$\Rightarrow \Delta \mathcal{P} = \frac{\epsilon_1 \epsilon_2 \sigma}{\epsilon_1 + \epsilon_2} [T_1^4 - T_2^4] \tag{21}$$