In this solution, unit system $h = 1 = k_B$ is used. One can restore it by dimensional analysis $\beta = \frac{1}{T}$

3.1 Vertical column of ideal gas

Canonical energy per particle is given by $\epsilon = T + U = \frac{p^2}{2m} + mgh$. We integrate through phase space for partition function Z_{1p} since energy is continuous.

$$\begin{aligned} Z_{1p} &= \int dV \int d^3\vec{p} e^{-\beta\epsilon} \\ &= \int dSdh \int d\vec{p} e^{-\beta\left[\frac{\vec{p}^2}{2m}+mgh\right]} \\ &= S\int_0^{h_0} dh\, e^{-\beta mgh} \prod_{i=1}^{3}\int_{-\infty}^{\infty} dp_i\, e^{-\beta\frac{p_i^2}{2m}} \\ &= S\frac{1}{\beta mg}\left[1-e^{-\beta mgh_0}\right]\left(\frac{2\pi m}{\beta}\right)^{3/2} \equiv \frac{ST}{mg}[1-e^{-mgh_0/T}]\left(\frac{2\pi mT}{h^2}\right)^{3/2} \end{aligned} \tag{1}$$

After obtaining partition function Z_{1p}, $Z = \frac{Z_{1p}^N}{N!}$, it's easy for us to find free energy F and internal energy E

$$\begin{aligned} F &= -\frac{1}{\beta}\log(Z) = -\frac{3N}{2\beta}\log\frac{2\pi m}{\beta} - \frac{N}{\beta}\log\frac{S(1-e^{-\beta mgh_0})}{N\beta mg} - \frac{N}{\beta} \\ E &= -\frac{\partial}{\partial\beta}\log(Z) = -N\frac{\partial}{\partial\beta}\log(Z_{1p}) = \frac{3N}{2}\frac{\partial}{\partial\beta}\log\beta + N\frac{\partial}{\partial\beta}\left[\log\beta - \log\left(1-e^{-\beta mgh_0}\right)\right] \\ &= \frac{5N}{2\beta} - \frac{Nmgh_0e^{-\beta mgh_0}}{1-e^{-\beta mgh_0}} = \frac{5N}{2\beta} - \frac{Nmgh_0}{e^{\beta mgh_0}-1} \\ h_{c.m.} &= \frac{\int Nmhe^{-\beta\epsilon}}{\int Nme^{-\beta\epsilon}} = -\frac{1}{\beta m}\frac{\partial}{\partial g}\log Z_{1p} = \frac{1}{\beta mg} - \frac{h_0}{e^{\beta mgh_0}-1} \\ C &= \frac{\partial E}{\partial T} = \frac{5N}{2} - Ne^{\beta mgh_0}\left[\frac{\beta mgh_0}{e^{\beta mgh_0}-1}\right]^2 \end{aligned} \tag{2}$$

Consider two cases. First $mgh_0 >> T \Rightarrow \beta mgh_0 >> 1 \Rightarrow e^{-\beta mgh_0} << 1$

$$h_{c.m.} = \frac{1}{\beta mg}\ ;\ C = \frac{5N}{2} \tag{3}$$

In another case $mgh_0 << T \Rightarrow \beta mgh_0 << 1 \Rightarrow e^{-\beta mgh_0} \simeq 1 - \beta mgh_0 + \frac{(\beta mgh_0)^2}{2}$

$$\begin{aligned} h_{c.m.} &\simeq \frac{1}{\beta mg} - \frac{h_0}{\beta mgh_0 + \frac{1}{2}(\beta mgh_0)^2} \simeq \frac{h_0}{2} \\ C &\simeq \frac{5N}{2} - N\left[\frac{\beta mgh_0}{\beta mgh_0}\right]^2 = \frac{3N}{2} \end{aligned} \tag{4}$$

3.2 Ultrarelativistic gas

Ultrarelativistic gas has $\epsilon = c\,|\vec{p}|$. It's continuous spectrum thus we need to integrate through phase space for a single paticle. $v = \frac{V}{N}$

$$\begin{aligned}
Z_{1p} &= \int dV \int d^3\vec{p}\, e^{-\beta\epsilon} = 4\pi V \int_0^\infty p^2 e^{-\beta c p} dp = 4\pi V \frac{2}{(\beta c)^3} \Rightarrow Z = \frac{Z_{1p}^N}{N!} \\
F &= \frac{-1}{\beta}\log Z = \frac{-N}{\beta}[\log(8\pi v) - 3\log(\beta c) + 1] \\
E &= -\frac{\partial}{\partial\beta}\log Z = \frac{3N}{\beta} \\
S &= \frac{E-F}{T} = \beta(E-F) = N[4 - 3\log(\beta c) + \log(8\pi v)] \\
p &= -\left(\frac{\partial F}{\partial V}\right)_\beta = \frac{1}{\beta}\frac{\partial}{\partial V}\log Z = \frac{1}{\beta}\frac{\partial}{\partial v}\log Z_{1p} = \frac{1}{\beta v}
\end{aligned} \tag{5}$$

In an adiabatic expension $dQ = 0 \Rightarrow dE = -pdv$. Express p, T in function of v

$$\begin{aligned}
&3dT = dE = -pdv = -\frac{Tdv}{v} \Rightarrow Tv^{\frac{1}{3}} = const \\
&T = pv \Rightarrow pv^{\frac{4}{3}} = const
\end{aligned} \tag{6}$$

3.3 Dipole molecules

For diatomic particle with permanent electric dipole moment $\vec{b}$ the total energy can be expressed as $\varepsilon = \varepsilon_{c.m.} + \varepsilon_{rot} + \varepsilon_{ang} = \frac{\vec{p}^2}{2m} + \frac{1}{2}\mathbb{I}_{ij}\omega_i\omega_j - \vec{b}\cdot\vec{\mathcal{E}}$. We can separate center of mass frame and rotation frame contribution $Z_{1p}(T,\vec{\mathcal{E}}) = Z_{c.m.}(T)Z_{rot}(T)Z_{ang}(T,\vec{\mathcal{E}})$. We are only interested in Z_{ang} in this case. Assume $\vec{\mathcal{E}} = \mathcal{E}\hat{z}$

$$Z_{ang} = \int d\Omega e^{\beta\vec{b}\cdot\vec{\mathcal{E}}} = 2\pi\int_{-\pi}^{\pi}\sin\theta d\theta e^{\beta b\mathcal{E}\cos\theta} = 2\pi\frac{2\sinh(\beta b\mathcal{E})}{\beta b\mathcal{E}} \tag{7}$$

Calculate angular probability distribution by

$$w = \frac{e^{-\beta b\mathcal{E}\cos\theta}}{Z_{ang}} \Rightarrow \frac{\partial w}{\partial\theta} = \frac{2\pi\beta b\mathcal{E}\sin\theta e^{\beta b\mathcal{E}\cos\theta}}{Z_{ang}} = \frac{(\beta b\mathcal{E})^2\sin\theta e^{\beta b\mathcal{E}\cos\theta}}{2\sinh\beta b\mathcal{E}} \tag{8}$$

(No points will be subtracted if 2π is not present. $\beta b\mathcal{E}$ will have power 1 after normalization.)
There are two ways to find polarization: one by definition, the other via energy

$$\begin{aligned}
&1.\ P(T,\mathcal{E}) = \frac{\sum_{\text{dipole}}\int d\Omega\, b\cos\theta e^{-\beta\epsilon_{ang}}}{\int d\Omega e^{-\beta\epsilon_{ang}}} = \frac{N}{\beta}\frac{\partial}{\partial\mathcal{E}}\log Z_{ang} = Nb\coth(\beta b\mathcal{E}) - \frac{N}{\beta\mathcal{E}} \\
&2.\ P(T,\mathcal{E}) = -\frac{E}{\mathcal{E}} = \frac{1}{\mathcal{E}}\frac{\partial}{\partial\beta}\log Z_{ang}^N = \frac{N}{\mathcal{E}}\left[b\mathcal{E}\coth(\beta b\mathcal{E}) - \frac{1}{\beta}\right]
\end{aligned} \tag{9}$$

In limit $T \to \infty$, $\Rightarrow \beta \to 0$

$$P(\beta\to 0,\mathcal{E}) = \lim_{\beta\to 0} Nb\frac{1+\frac{(\beta b\mathcal{E})^2}{2}}{\beta b\mathcal{E}} - \frac{N}{\beta\mathcal{E}} = 0 \tag{10}$$

As we know in high temperature, thermal fluctuation causes dipoles pointing in random direction, leaving material unpolarized. We may also check in $T \to 0$, $\beta \to \infty$ limit, $P(\beta\to\infty,\mathcal{E}) = Nb$. This also matchs our expectation. (Maximally polarized)

3.4 Paramagnetic medium II

N non-interacting magnetic dipoles. $\mu_z = \frac{m}{J}\mu$, where $-J \le m \le J$. Energy in an external magnetic field $\epsilon = -\frac{m}{J}\mu\mathcal{H}$.
(a)
Partition function

$$Z_N(T,\mathcal{H}) = \sum_{\text{state}} e^{-\beta\epsilon} = \frac{1}{N!}\left[\sum_{m=-J}^{J} e^{\beta\frac{m}{J}\mu\mathcal{H}}\right]^N = \frac{1}{N!}\left[\frac{e^{-\beta\mu\mathcal{H}}(1-e^{\beta\frac{2J+1}{J}\mu\mathcal{H}})}{1-e^{\beta\frac{1}{J}\mu\mathcal{H}}}\right]^N = \frac{1}{N!}\left[\frac{\sinh\beta\frac{2J+1}{2J}\mu\mathcal{H}}{\sinh\beta\frac{1}{2J}\mu\mathcal{H}}\right]^N \tag{11}$$

As $J=\frac{1}{2}$ (two level spin), we get the same result as HW 2.4.
(b)

$$\begin{aligned} F &= -\frac{1}{\beta}\log Z = N\left[\log N - 1 - \log\sinh\beta\frac{2J+1}{2J}\mu\mathcal{H} + \log\sinh\beta\frac{1}{2J}\mu\mathcal{H}\right] \\ E &= -\frac{\partial}{\partial\beta}\log Z = -N\mu\mathcal{H}\left[\frac{2J+1}{2J}\coth\beta\frac{2J+1}{2J}\mu\mathcal{H} - \frac{1}{2J}\coth\beta\frac{1}{2J}\mu\mathcal{H}\right] \\ M &= -\frac{E}{\mathcal{H}} = N\mu\left[\frac{2J+1}{2J}\coth\beta\frac{2J+1}{2J}\mu\mathcal{H} - \frac{1}{2J}\coth\beta\frac{1}{2J}\mu\mathcal{H}\right] \\ \chi_T &= \left(\frac{\partial M}{\partial\mathcal{H}}\right)_T = \frac{N\beta\mu^2}{4J^2}\left[\frac{1}{\sinh^2\beta\frac{1}{2J}\mu\mathcal{H}} - \frac{(2J+1)^2}{\sinh^2\beta\frac{2J+1}{2J}\mu\mathcal{H}}\right] \\ \left(\frac{\partial M}{\partial T}\right)_{\mathcal{H}} &= -\frac{N\mu^2\mathcal{H}\beta^2}{4J^2}\left[\frac{1}{\sinh^2\beta\frac{1}{2J}\mu\mathcal{H}} - \frac{(2J+1)^2}{\sinh^2\beta\frac{2J+1}{2J}\mu\mathcal{H}}\right] \end{aligned} \tag{12}$$

(c)

$$C_{\mathcal{H}} = \left(\frac{\partial E}{\partial T}\right)_{\mathcal{H}} = \frac{d\beta}{dT}\left(\frac{\partial E}{\partial\beta}\right)_{\mathcal{H}} = \frac{\beta^2 N\mu^2\mathcal{H}^2}{4J^2}\left[\frac{(2J+1)^2}{\sinh^2\beta\frac{2J+1}{2J}\mu\mathcal{H}} - \frac{1}{\sinh^2\beta\frac{1}{2J}\mu\mathcal{H}}\right] \tag{13}$$