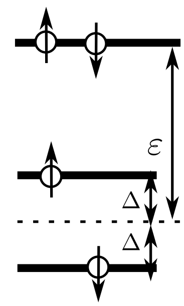
PHY 540 Statistical Mechanics

Midterm exam, October 21, 2021

## 1. Two fermions and fine structure(25)




A single fermion with spin $\sigma_z = \pm\frac{1}{2}$ can occupy either (1) one of the "fine-structure" levels with spin-dependent energy $\pm\Delta$, or (2) a degenerate excited level with spin-independent energy $\varepsilon$. Below, we consider two fermions coexisting in these energy levels and constrained by the Pauli exclusion principle.

**(a)(7)** Enumerate all energy and spin states of this **two-fermion** system.

**(b)(8)** Calculate the partition function, the total internal energy, and the entropy of the system as functions of temperature $T$.

**(c)(10)** Calculate the total polarization $\langle\sigma_z\rangle = \langle\sigma_z^{(1)} + \sigma_z^{(2)}\rangle$ and the "susceptibility" $\left(\frac{\partial\langle\sigma_z\rangle}{\partial\Delta}\right)_T$.

## 2. 1D Fermionic gas in a harmonic trap(25)

One-dimensional gas of $N \gg 1$ non-interacting spin-$\frac{1}{2}$ fermions is confined by a harmonic potential, so that the energy per particle is

$$\varepsilon(p, x) = \frac{p^2}{2m} + \frac{1}{2}\kappa x^2 \,.$$

Assume that the temperature is small $T \ll \varepsilon_F$ compared to the Fermi energy, but large compared to the harmonic level spacing $T \gg \hbar\omega = \hbar\sqrt{\kappa/m}$.

**(a)(7)** Determine the Fermi energy $\varepsilon_F$ and calculate the total energy of the gas $E_0 = E(T = 0)$ at zero temperature.

**(b)(8)** Find the local number density of fermions $n(x) = \frac{dN}{dx}$.

**(c)(10)** Calculate magnetic susceptibility of the gas if each fermion has magnetic moment $\pm\mu$ and corresponding additional energy $\Delta\varepsilon = \mp\mu H$ in a magnetic field $H$.