1. Two fermions and fine structure(25)

(a) Taking into account the Pauli exclusion principle, the following states with the total energy E and spin σ_z are possible:

- 1 level with $E = 0$, $\sigma_z = 0$;
- 2 levels with $E = \epsilon - \Delta$, $\sigma_z = -1$ or 0;
- 2 levels with $E = \epsilon + \Delta$, $\sigma_z = +1$ or 0;
- 1 level with $E = 2\epsilon$, $\sigma_z = 0$.

(b) the partition function is a sum over all distinct energy states of the entire system,

$$Z = 1 + 2e^{-\beta(\varepsilon-\Delta)} + 2e^{-\beta(\varepsilon+\Delta)} + e^{-2\beta\varepsilon} \tag{1}$$

and the energy can be computed either as $-\left(\frac{\partial \log Z}{\partial \beta}\right)$ or using the probabilities $w = Z^{-1}e^{-\beta E}$:

$$\begin{aligned}\langle E\rangle &= Z^{-1}\big(0\cdot 1 + 2(\varepsilon-\Delta)\cdot e^{-\beta(\varepsilon-\Delta)} + 2(\varepsilon+\Delta)\cdot e^{-\beta(\varepsilon+\Delta)} + 2\varepsilon e^{-2\beta\varepsilon}\big) \\ &= \frac{4e^{-\beta\varepsilon}\big[\varepsilon\cosh(\beta\Delta) + \Delta\sin(\beta\Delta)\big] + 2\varepsilon e^{-2\beta\varepsilon}}{1 + 2e^{-\beta(\varepsilon-\Delta)} + 2e^{-\beta(\varepsilon+\Delta)} + e^{-2\beta\varepsilon}}\,.\end{aligned} \tag{2}$$

(c) The polarization is also easily computed using the Gibbs distribution (only the states with non-vanishing total spin contribute):

$$\langle\sigma_z\rangle = Z^{-1}\big[(+1)\cdot e^{-\beta(\varepsilon+\Delta)} + (-1)\cdot e^{-\beta(\varepsilon-\Delta)}\big] = \frac{e^{-\beta\varepsilon}\cdot 2\sinh(\beta\Delta)}{1 + 2e^{-\beta(\varepsilon-\Delta)} + 2e^{-\beta(\varepsilon+\Delta)} + e^{-2\beta\varepsilon}}\,. \tag{3}$$

2. 1D Fermionic gas in a harmonic trap(25)

(a)(5) At $T = 0$, all levels $\varepsilon \le \varepsilon_F$ are filled and contain $N = g_s\frac{\varepsilon_F}{\hbar\omega}$, where $g_s = 2$ is the spin degeneracy. Therefore, the Fermi energy is $\varepsilon_F = \hbar\omega\frac{N}{g_s}$. Alternatively, the number of fermions can be determined by integrating over the phase space

$$N = g_s\int_{\varepsilon\le\varepsilon_F}\frac{dx\,dp}{2\pi\hbar} = \frac{g_s}{2\pi\hbar}\sqrt{\frac{m}{\kappa}}\int_{\tilde{x}^2+\tilde{p}^2\le 2\varepsilon_F} d\tilde{x}\,d\tilde{p} = \frac{g_s\varepsilon_F}{\hbar\omega} \tag{4}$$

which leads to the same answer.

(b)(5) The spatial density can be obtained similarly to the previous part, except integrating only over momenta constrained by the energy $|p| \leq \sqrt{2m(\varepsilon_F - \frac{1}{2}\kappa x^2)}$:

$$n(x) = g_s \int_{\varepsilon \leq \varepsilon_F} \frac{dp}{2\pi\hbar} = \frac{g_s}{2\pi\hbar}\, 2\sqrt{2m(\varepsilon_F - \frac{1}{2}\kappa x^2)}\,. \tag{5}$$

(c)(5) The density of states is computed as

$$g(\varepsilon_F) = \frac{dN(\varepsilon_F)}{d\varepsilon_F} = \frac{g_s}{\hbar\omega}\,. \tag{6}$$

In a small magnetic field, the magnetization is

$$M = \mu(N_+ - N_-) \approx (+\mu) \cdot \frac{1}{2}(N + g(\varepsilon_F)\mu H) + (-\mu) \cdot \frac{1}{2}(N - g(\varepsilon_F)\mu H) = \mu^2 g(\varepsilon_F) H \tag{7}$$

and the magnetic susceptibility $\chi = \frac{g_s \mu^2}{\hbar\omega}$.

(d)(10) The chemical potential does not change for small temperature, because the levels are equidistant, the level density $g(\varepsilon) = \frac{g_s}{\hbar\omega}$ is constant, and at small temperatures $T \ll \varepsilon_F$ the fermions are redistributed symmetrically towards higher and lower energies. This can be strictly shown using Sommerfeld expansion of the particle number

$$N = \int_0^\infty \frac{d\varepsilon\, g(\varepsilon)}{e^{(\varepsilon-\mu)/T} + 1} \approx \int_0^\mu d\varepsilon\, g(\varepsilon) + \frac{1}{6}\pi^2 T^2 g'(\varepsilon) \approx N + (\mu - \varepsilon_F) g(\varepsilon) + \frac{1}{6}\pi^2 T^2 g'(\varepsilon) \tag{8}$$

and thus $\mu - \varepsilon_F \approx \frac{1}{6}\pi^2 T^2 \frac{g'(\varepsilon)}{g(\varepsilon)} = 0$ for the harmonic potential.

To calculate the total energy of the degenerate fermion gas, the Sommerfeld expansion is used again

$$E = \int_0^\infty \frac{d\varepsilon\, \varepsilon\, g(\varepsilon)}{e^{(\varepsilon-\mu)/T} + 1} \approx E_0 + (\mu - \varepsilon_F)\varepsilon g(\varepsilon) + \frac{1}{6}\pi^2 T^2 (\varepsilon g)' = E_0 + \frac{1}{6}\pi^2 T^2 g(\varepsilon) \tag{9}$$

and in this case $C = \frac{\pi^2 g_s T}{3\hbar\omega}$.