1.1 Derivative of metric determinant

Derivatives of metric determinant

$$\frac{1}{g}\partial_\mu g = g^{\rho\sigma}\partial_\mu g_{\rho\sigma}\,, \quad g = \det||g_\mu\nu||\,, \quad ||g^{\mu\nu}|| = ||g_{\mu\nu}||^{-1}\,, \tag{1}$$

can be most transparently derived by differentiating the matrix identity

$$\log\det A = \mathrm{Tr}\log A\,, \tag{2}$$

as follows (λ can be any parameter, not necessarily a coordinate)

$$\frac{d}{d\lambda}(\log\det A) = \frac{1}{\det A}\frac{d\det A}{d\lambda} = \left(\frac{1}{\Delta\lambda}\mathrm{Tr}\big[\log(A+\Delta A) - \log A\big]\right)_{\Delta\lambda\to 0}\,. \tag{3}$$

To evaluate the difference of the logs up to $O(\Delta A)$, one can use the identity (Baker-Campbell-Hausdorff formula)

$$e^X\cdot e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\ldots} \tag{4}$$

that can be verified through Taylor expansion. The higher-order terms indicated by $\ldots$ consist of terms X^aY^{n-a} with $a = 1\ldots(n-1)$ and contain terms linear in X and Y. However, they all are commutators and vanish when the trace is taken, so rewriting the identity with $e^X = A+\Delta A$, $e^Y = A^{-1}$, one gets

$$\begin{aligned}\log(A+\Delta A) - \log A &= \log\left[(A+\Delta A)\cdot A^{-1}\right] + \text{commutators}\,,\\ \mathrm{Tr}\big[\log(A+\Delta A)-\log A\big] &= \mathrm{Tr}\log\left[1+\Delta A\cdot A^{-1}\right] = \mathrm{Tr}\big[A^{-1}\cdot\Delta A\big] + O\big((\Delta A)^2\big)\end{aligned} \tag{5}$$

and therefore

$$\frac{d}{d\lambda}(\log\det A) = \mathrm{Tr}\left[A^{-1}\cdot\frac{dA}{d\lambda}\right] \tag{6}$$

1.2 Alternative derivation

The equation (1) can be derived without relying on the identity (2). Instead, one can write explicit formulas for the determinant and the inverse of an arbitrary nondegenerate matrix A as

$$\det A = \sum_i(-1)^{i+j}A_{ij}M_{ij}\,, \tag{7}$$

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A}(-1)^{i+j}M_{ji}\,, \tag{8}$$

where M_{ij} is the minor determinant of matrix A obtained by crossing out row i and column j. The first equation (7) is the standard definition of the determinant, and the second equation can be easily verified by observing that

$$\sum_i(-1)^{i+j}M_{ij}A_{ik} = \delta_{jk}\,\det A\,, \tag{9}$$

which follows directly from Eq. (7) for $j = k$. For $j\neq k$, the lhs of Eq.(9) is equal to the determinant of the matrix A with column $A_{.j}$ replaced with $A_{.k}$, which is zero because such matrix has two identical columns. When differentiating Eq.(7), one has to take into account that each matrix element can depend on parameter λ, therefore

$$\frac{d}{d\lambda}\det A = \sum_{ij}\frac{\partial\det A}{\partial A_{ij}}\frac{dA_{ij}}{d\lambda} \tag{10}$$

and the partial derivatives can be expressed through A^{-1} using Eqn. (8),

$$\frac{\partial\det A}{\partial A_{ij}} = (-1)^{i+j}M_{ij} = \det A\big(A^{-1}\big)_{ji}\,, \tag{11}$$

hence

$$\frac{1}{\det A}\frac{d}{d\lambda}\det A = \sum_{ij}\left(A^{-1}\right)_{ji}\frac{dA_{ij}}{d\lambda} = \mathrm{Tr}\left[A^{-1}\cdot\frac{dA}{d\lambda}\right] \tag{12}$$